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NOTE

A Schwarz Lemma for Convex

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In this note the following new version of the Schwarz lemma is proved: If f is a holomorphic function mapping a bounded convex domain $\mathscr{D}_1$ of a complex Banach space into a convex domain $\mathscr{D}_2$ of another complex Banach space and $f(a) = b$, then the image by f of the set of points in $\mathscr{D}_1$ lying at a distance greater than r from the frontier of $\mathscr{D}_1$ is at a positive distance from the frontier of $\mathscr{D}_2$. This distance depends only upon a, b, and r, and it can be estimated specifically in terms of the norms of the Banach spaces. Our result extends several earlier theorems.

1. INTRODUCTION AND NOTATION

After H. A. Schwarz proved his famous lemma in 1869 (see, for instance, [5, pp. 235–241]), this subject has been extensively developed and generalized. G. Pick (1916) extended the lemma by removing the hypothesis that the origin is a fixed point. In fact, he showed that every analytic function $f : \mathbb{D} \to \mathbb{D}$ is nonexpansive for the Poincaré distance ρ on the open unit

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disk $\mathbb{D}$ of the complex plane $\mathbb{C}$. That is, $\rho(f(z), f(w)) \leq \rho(z, w)$ for all $z, w \in \mathbb{D}$, where

$$\rho(z, w) = \tanh^{-1}\left|\frac{z-w}{1-\bar{z}w}\right|.$$

If $(E, \|\cdot\|)$ is a normed linear space, then B_E will stand for its open unit ball and $B(a, \varepsilon)$ $(a \in E, \varepsilon > 0)$ is the open ball $\{x \in E : \|x - a\| < \varepsilon\}$. A domain in E is a nonempty connected open subset. If A, B are subsets of E then $\operatorname{dist}(A, B)$ denotes, as usual, the infimum of $\|x - y\|$ $(x \in A, y \in B)$. If $x, y \in E$ then $[x, y]$ denotes the closed segment joining x to y. If $\mathscr{D}_1, \mathscr{D}_2$ are domains in two complex Banach spaces E, F, respectively, then we denote by $H(\mathscr{D}_1, \mathscr{D}_2)$ the set of all holomorphic mappings from $\mathscr{D}_1$ into $\mathscr{D}_2$. The Graves–Taylor–Hille–Zorn's theorem asserts (see, for instance, [1, pp. 196–203]) that $f \in H(\mathscr{D}_1, \mathscr{D}_2)$ if and only if f is continuous and, for all $a \in \mathscr{D}_1$, $x \in E$, and $\lambda \in F^*$, the function $z \in U(a, x) \mapsto \lambda(f(a + zx)) \in \mathbb{C}$ is holomorphic, where F^* is the topological dual space of F and $U(a, x) = \{z \in \mathbb{C} : a + zx \in \mathscr{D}_1\}$. In 1969 Harris [4] extended the Schwarz lemma to unit balls of arbitrary complex Banach spaces (see also [3, III.2.3]) and, in 1973 Renaud [8] studied the subject in the setting of Hilbert spaces.

If $\mathscr{D}$ is a domain in a Banach space E, the mapping

$$\begin{aligned} C = C_{\mathscr{D}} : (x, y) &\mapsto C(x, y) \\ &= \sup\{\rho(f(x), f(y)) : f \in H(\mathscr{D}, \mathbb{D})\} \in [0, +\infty) \end{aligned}$$

is called the *Carathéodory pseudodistance* on E. It satisfies $C_{\mathbb{D}} = \rho$ and the Schwarz–Pick inequality: if $f \in H(\mathscr{D}_1, \mathscr{D}_2)$, then

$$C_{\mathscr{D}_2}(f(x), f(y)) \leq C_{\mathscr{D}_1}(x, y) \qquad (\forall x, y \in \mathscr{D}_1). \tag{1}$$

For this, see [2, pp. 47–48]. Let us point out here that [2] is a very good source for a review of the present development of the subject relationship to (or application on) potential theory, fixed point theory, differential geometry, Radon–Nikodym property, automorphisms of complex manifolds, etc.

We are interested in providing a version of the Schwarz lemma *in terms of the norms* of two Banach spaces E and F. Denote by A^c the complement of a set A and by $\|\cdot\|$ the norm either on E or on F, without distinction. We propose the following question:

> How far is a holomorphic image of a subdomain of a domain in E from the frontier of the range domain?

By rephrasing the classical Schwarz lemma we have

If $\Phi \in H(\mathbb{D}, \mathbb{D})$ and $\Phi(0) = 0$, then $\operatorname{dist}(\Phi(r\mathbb{D}), \mathbb{D}^c) \geq 1 - r$ for all $r \in (0,1)$. In [3, III.2.3] the same is proved for two arbitrary Banach spaces, that is,

If $\Phi \in H(B_E, B_F)$ and $\Phi(0) = 0$, then $\operatorname{dist}(\Phi(rB_E), B_F^c) \geq 1 - r$ for all $r \in (0,1)$.

Recently, Prieto [7] has generalized this result by proving that for each $r \in (0,1)$ and each $b \in B_F$ there exists a real number $m = m(r, \|b\|) \in (0,1)$ satisfying the following:

If $\Phi \in H(B_E, B_F)$, $\Phi(0) = b$, and $z \in rB_E$, then $\|\Phi(z)\| \leq m$, that is, $\operatorname{dist}(\Phi(rB_E), B_F^c) \geq 1 - m$. Specifically, $m(r, \|b\|) = (\|b\| + r)/(1 + \|b\|r)$.

In this note we improve all the above statements. In fact, we provide a new version of the Schwarz lemma by considering two convex domains $\mathscr{D}_1, \mathscr{D}_2$, instead of the balls B_E, B_F.

2. SOME PRELIMINARY RESULTS

In order to prove our theorem, we will make use of two lemmas. The first one is purely geometric. The proof of the second one can be found in [6, pp. 36–37].

LEMMA 1. *Let $\mathscr{D} \subset E$ be a convex domain, $r > 0$, and $a \in \mathscr{D}$. Denote by $T(r,a)$ the set*

$$T(r,a) = \bigcup \{[a,z]\colon B(z,r) \subset \mathscr{D}\}.$$

Then

$$\operatorname{dist}(T(r,a), \mathscr{D}^c) \geq \min(r, \operatorname{dist}(a, \mathscr{D}^c)). \tag{2}$$

Proof. Fix $x \in T(r,a)$ and $\varepsilon > 0$ with $\varepsilon < r$, $\varepsilon < \operatorname{dist}(a, \mathscr{D}^c)$. Then $B(a,\varepsilon) \subset \mathscr{D}$ and there exist $t \in [0,1]$ and $z \in \mathscr{D}$ such that $x = at + (1-t)z$ and $B(z,r) \subset \mathscr{D}$. We must show that $B(x,\varepsilon) \subset \mathscr{D}$. Indeed, if $y \in B(x,\varepsilon)$, then $y = x + v$, where $\|v\| < \varepsilon$. Thus $y = at + (1-t)z + v = t(a+v) + (1-t)(z+v)$. But $a + v \in B(a,\varepsilon)$ and $z + v \in B(z,r)$, so the points $a+v, z+v$ are in $\mathscr{D}$. From the convexity of $\mathscr{D}$ we obtain $y \in \mathscr{D}$ for all $y \in B(x,\varepsilon)$. ∎

LEMMA 2. *Let $\mathscr{D} \subset E$ a bounded domain and assume that $K \subset \mathscr{D}$ is a convex set with $\delta =: \operatorname{dist}(K, \mathscr{D}^c) > 0$. Then*

$$C_{\mathscr{D}}(x,y) \leq \frac{1}{\delta}\|x - y\| \tag{3}$$

for all $x, y \in K$.

3. A NEW VERSION OF THE SCHWARZ LEMMA

Next, we state our theorem.

THEOREM. *Let E, F be two complex Banach spaces. Assume that $\mathscr{D}_1 \subset E$ and $\mathscr{D}_2 \subset F$ are convex domains and that $\mathscr{D}_1$ is bounded. Fix two points $a \in \mathscr{D}_1$, $b \in \mathscr{D}_2$ and a real number $r > 0$. Then there exists a real number $s = s(a, b, r) > 0$ satisfying*

$$\operatorname{dist}\big(\Phi\big(\{z \in \mathscr{D}_1 : d(z, \mathscr{D}_1^{\mathrm{c}}) > r\}\big), \mathscr{D}_2^{\mathrm{c}}\big) \geq s$$

for all $\Phi \in H(\mathscr{D}_1, \mathscr{D}_2)$ with $\Phi(a) = b$. Specifically,

$$s(a, b, r) = \operatorname{dist}(b, \mathscr{D}_2^{\mathrm{c}}) \exp\left(- \frac{2\mu(a)}{\min\big(r, \operatorname{dist}(a, \mathscr{D}_1^{\mathrm{c}})\big)} \right),$$

where $\mu(a) = \sup\{\|z - a\| : z \in \mathscr{D}_1\}$.

Proof. Denote by S the following subset of $\mathscr{D}_2$:

$$S = \big\{\Phi(z) : \operatorname{dist}(z, \mathscr{D}_1^{\mathrm{c}}) > r, \Phi \in H(\mathscr{D}_1, \mathscr{D}_2), \Phi(a) = b\big\}.$$

By hypothesis, the set $\mathscr{D}_2 - b = \{w - b : w \in \mathscr{D}_2\}$ is convex. Let $\varphi \in F^*$ and suppose

$$\sup_{w \in \mathscr{D}_2 - b} \operatorname{Re} \varphi(w) \leq 1. \tag{4}$$

Define $g(w) = \varphi(w - b)/(2 - \varphi(w - b))$ ($\forall w \in \mathscr{D}_2$). Then $g(b) = 0$ and $|g(w)| \leq 1$ for all $w \in \mathscr{D}_2$. From the maximum modulus principle [1, pp. 180–181] we obtain that $g \in H(\mathscr{D}_2, \mathbb{D})$. If $w \in \mathscr{D}_2$, we have that

$$\begin{aligned} \tanh^{-1}|g(w)| &= \tanh^{-1}\left| \frac{g(w) - g(b)}{1 - \overline{g(b)}g(w)} \right| \\ &= \rho(g(b), g(w)) \leq C_{\mathscr{D}_2}(b, w). \end{aligned}$$

If $w \in S$ then there exist $z \in \mathscr{D}_1$ and $\Phi \in H(\mathscr{D}_1, \mathscr{D}_2)$ such that $\operatorname{dist}(z, \mathscr{D}_1^{\mathrm{c}}) > r$, $\Phi(a) = b$, and $\Phi(z) = w$. Hence, from (1),

$$\tanh^{-1}|g(w)| \leq C_{\mathscr{D}_2}(\Phi(a), \Phi(z)) \leq C_{\mathscr{D}_1}(a, z).$$

Now, we can apply Lemma 2 on $\mathscr{D} = \mathscr{D}_1$ and $K = [a, z]$. We get from (3) that

$$C_{\mathscr{D}_1}(a, z) \leq \frac{\|z - a\|}{\operatorname{dist}([a, z], \mathscr{D}_1^c)}.$$

Since $B(z, r) \subset \mathscr{D}_1$, (2) in Lemma 1 can be used to obtain

$$\operatorname{dist}([a, z], \mathscr{D}_1^c) \geq \operatorname{dist}(T(r, a), \mathscr{D}_1^c) \geq \min(r, \operatorname{dist}(a, \mathscr{D}_1^c)).$$

Thus

$$\begin{aligned} \tanh^{-1}|g(w)| &\leq \frac{\|z - a\|}{\min(r, \operatorname{dist}(a, \mathscr{D}_1^c))} \\ &\leq \frac{\mu(a)}{\min(r, \operatorname{dist}(a, \mathscr{D}_1^c))} \equiv t \end{aligned}$$

and

$$|g(w)| \leq \tanh t \equiv \alpha$$

or, equivalently,

$$\left| \frac{\varphi(w - b)}{2 - \varphi(w - b)} \right| \leq \alpha$$

for all $w \in S$. If we put $\varphi(w - b) = u + iv$ then $(u^2 + v^2)/((2 - u)^2 + v^2) \leq \alpha^2$, so $(1 - \alpha^2)u^2 + 4\alpha^2 u - 4\alpha^2 \leq v^2(\alpha^2 - 1) \leq 0$, because $\alpha < 1$. Thus $(u - 2\alpha/(1 + \alpha))(u + 2\alpha/(1 - \alpha)) \leq 0$, and, therefore,

$$\operatorname{Re} \varphi(w - b) \leq \beta \qquad (\forall w \in S), \tag{5}$$

where $\beta = 2\alpha/(1 + \alpha)$. Note that $\beta < 1$, because $\alpha < 1$. Choose $\gamma \in (0, \operatorname{dist}(b, \mathscr{D}_2^c))$, so that $\gamma B_F \subset \mathscr{D}_2 - b$.

We now use (4) and (5) to get that, for all $w \in S - b + \gamma(1 - \beta)B_F$,

$$\begin{aligned} \operatorname{Re} \varphi(w) &\leq \beta + \gamma(1 - \beta) \sup_{w \in B_F} \operatorname{Re} \varphi(w) = \beta + (1 - \beta) \sup_{w \in \gamma B_F} \operatorname{Re} \varphi(w) \\ &\leq \beta + (1 - \beta) \sup_{w \in \mathscr{D}_2 - b} \operatorname{Re} \varphi(w) \leq \beta + 1 - \beta = 1. \end{aligned}$$

We summarize: If φ is in F^* and satisfies (4), then $\operatorname{Re} \varphi(w) \leq 1$ for all $w \in S - b + \gamma(1 - \beta)B_F$. Since $\mathscr{D}_2 - b$ is convex, the Hahn–Banach theorem says that $S - b + \gamma(1 - \beta)B_F \subset \mathscr{D}_2 - b$ or, equivalently, $S +$

$\gamma(1-\beta)B_F \subset \mathscr{D}_2$. Hence $\operatorname{dist}(S, \mathscr{D}_2^c) \geq \gamma(1-\beta)$ for all $\gamma \in (0, \operatorname{dist}(b, \mathscr{D}_2^c))$. But,

$$\gamma(1-\beta) = \gamma\left(1 - \frac{2\alpha}{1+\alpha}\right) = \gamma\frac{1-\tanh t}{1+\tanh t}$$
$$= \gamma\frac{1-(e^{2t}-1)/(e^{2t}+1)}{1+(e^{2t}-1)/(e^{2t}+1)} = \gamma e^{-2t}.$$

Thus

$$\operatorname{dist}(S, \mathscr{D}_2^c) \geq \operatorname{dist}(b, \mathscr{D}_2^c)\exp\left(-\frac{2\mu(a)}{\min(r, \operatorname{dist}(a, \mathscr{D}_1^c))}\right) \equiv s(a,b,r).$$

as required. ∎

EXAMPLE. Consider the Banach space l_1 of complex sequences $z = \{z_n\}_1^\infty$ with $\|z\| = \sum_1^\infty |z_n| < +\infty$. If $\{\lambda_n\}_1^\infty$ is a sequence of positive real numbers then we denote by $\mathscr{R} = \mathscr{R}(\{\lambda_n\}_1^\infty)$ the infinite dimensional "rhombus" in l_1 with "center" at the origin and "semidiagonals" λ_n $(n \geq 1)$, that is,

$$\mathscr{R} = \left\{z = \{z_n\}_1^\infty : \sum_1^\infty \frac{|z_n|}{\lambda_n} < 1\right\}.$$

Its "vertices" are the points $\lambda_n e_n$ $(n \geq 1)$, where $\{e_n\}_1^\infty$ is the standard basic sequence of l_1. Trivially, $\mathscr{R}$ is a convex subset of l_1. It is an easy exercise to check that $\mathscr{R}$ is a bounded domain if and only if $0 < \alpha \leq \beta < +\infty$, where we have set

$$\alpha = \inf_n \lambda_n, \qquad \beta = \sup_n \lambda_n.$$

From now on we assume $0 < \alpha \leq \beta < +\infty$. A straightforward reasoning yields $\operatorname{dist}(0, \mathscr{R}^c) = \alpha$ and $\sup\{\|z\| : z \in \mathscr{R}\} = \beta$. Our result proves that the image of each vertex of the r-homothetic rhombus $\mathscr{H} = \mathscr{R}(\{r\lambda_n\}_1^\infty)$ $(0 < r < 1)$ by an arbitrary holomorphic self-map Φ on $\mathscr{R}$ fixing the origin is at least at a distance $\alpha\exp(-2\beta/\alpha(1-r))$ from any of the vertices of $\mathscr{R}$. Indeed, if $p, q \in \{1, 2, \ldots\}$, then $r\lambda_p e_p$ is in the closure of $\mathscr{H}$ and $\lambda_q e_q$ is in $\mathscr{R}^c$, so

$$\|\Phi(r\lambda_p e_p) - \lambda_q e_q\| \geq \operatorname{dist}(\Phi(\mathscr{H}), \mathscr{R}^c).$$

But $\operatorname{dist}(z, \mathscr{R}^c) > \alpha(1-r)$ for all $z \in \mathscr{H}$. Since $\Phi(0) = 0$ we derive that

$$\operatorname{dist}(\Phi(\mathscr{H}), \mathscr{R}^c) \geq \operatorname{dist}(0, \mathscr{R}^c) \exp\left(-\frac{2\sup\{\|z\| : z \in \mathscr{R}\}}{\min(\alpha(1-r), \operatorname{dist}(0, \mathscr{R}^c))}\right).$$

Consequently,

$$\|\Phi(r\lambda_p e_p) - \lambda_q e_q\| \geq \alpha \exp\left(-\frac{2\beta}{\alpha(1-r)}\right).$$

REFERENCES

1. S. B. Chae, ''Holomorphy and Calculus in Normed Spaces,'' Dekker, New York, 1985.
2. S. Dineen, ''The Schwarz Lemma,'' Clarendon Press, Oxford, 1989.
3. T. Franzoni and T. Vesentini, Holomorphic maps and invariant distances, *Math. Stud.*, Vol. 40, North Holland, Amsterdam, 1980.
4. L. A. Harris, Schwarz's lemma in normed linear spaces, *Proc. Nat. Acad. Sci. U.S.A.* **62** (1969), 1014–1017.
5. E. Hille, ''Analytic Function Theory,'' Vol. II, Chelsea, New York, 1973.
6. J. M. Isidro and L. L. Stachó, Holomorphic automorphic groups in Banach spaces, *Math. Stud.*, vol. 105, North Holland, Amsterdam, 1984.
7. A. Prieto, Sur le lemme de Schwarz en dimension infinie, *C. R. Acad. Sci. Paris. Sér. I* **314** (1992), 741–742.
8. A. Renaud, Quelques proprietés des applications analytiques d'une boule de dimension infinie dans une autre, *Bull. Sci. Math.*, (2) **97** (1973), 127–159.